PBW Bases for $U_q(\mathfrak{gl}(m|1))$

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1 The Classical Case

Let $\mathfrak{g}$ be a lie algebra and let $\alpha \in \Pi$. Let V be locally finite $\mathfrak{g}-$module(action of $e_{\pm\alpha}^N v = 0\ \forall v \in V$ for some N). Then

$$s_\alpha = \exp(e_\alpha)\exp(-f_\alpha)\exp(e_\alpha)$$

gives an automorphism of V as a v.s.

Lemma 1.1. *(a)* $s_\alpha(V_\mu) = V_{s_\alpha(\mu)}$

(b) $\langle s_\alpha \rangle_{\alpha\in\Pi}$ *gives an action of* B_W *on* V.

(c) $s_\alpha(X \cdot v) = s_\alpha(X) \cdot s_\alpha(v)$ *for all* $X \in U(\mathfrak{g})$.

Proof. (c) $U(\mathfrak{g})$ under the adjoint action is locally finite because on generators

- $\mathrm{ad}_{e_i}(e_i) = 0$
- $\mathrm{ad}_{e_i}^{1-a_{ij}}(e_j) = 0\ i \neq j$
- $\mathrm{ad}_{e_i}(h_j) = a_{ij}e_i$
- $\mathrm{ad}_{e_i}(f_j) = \delta_{ij}h_i$

and ad_{e_i} acts by derivations on $U(\mathfrak{g})$. Thus s_α gives an automorphism of $U(\mathfrak{g})$ as a vs. ■

Remark. $s_\alpha : U(\mathfrak{g}) \to U(\mathfrak{g})$ is an algebra homomorphism! Indeed one can compute

$$\mathrm{ad}_{s_\alpha(e_i)}(s_\alpha(h_j)) = s_\alpha(e_i)\cdot s_\alpha(h_j) \overset{(c)}{=\!=} s_\alpha(\mathrm{ad}_{e_i}(h_j)) = a_{ij}s_\alpha(e_i)$$

and similarly with the other relations.

Now when we move to $U_q(\mathfrak{g})$ we will have that $U_q(\mathfrak{g})$ is no longer locally finite under the adjoint action anymore. Indeed we have that

$$\mathrm{ad}_{e_i}^q(e_i) = e_i^2 - q^2 e_i^2 \neq 0$$

Thus the formula for s_α above will not give a map $U_q(\mathfrak{g}) \to U_q(\mathfrak{g})$. However Lusztig was nevertheless able to define maps T_α satisfying the conditions in the lemma above.

1.1 The Braid Group Action

The idea here is to reverse the flow of logic, to start from (c) and work our way back up to the definition. Let V be a f.d. $U_q(\mathfrak{sl}_2)-$module and $v \in V_m$. Then define

$$T(v) = \sum_{a,b,c\geq 0; -a+b-c=m} (-1)^b q^{b-ac} E^{(a)}F^{(b)}E^{(c)}v$$

Note $T : V_{q^m} \to V_{q^{-m}}$ (set $a = c = 0$).

Remark. $T(v)$ is just the quantum version of the action of s_α on v. Thus T is a bijection on V (switch the roles of E, F, etc).

In general for $v \in V_\mu$ we have

$$T \to T_\alpha, \quad E, F \to E_\alpha, F_\alpha, \quad q \to q_\alpha, \quad m \to \langle \mu, \alpha^\vee \rangle$$

Note that $T_\alpha : V_\mu \to V_{\mu - \langle \mu, \alpha^\vee \rangle \alpha} = V_{s_\alpha(\mu)}$ and thus why we chose m as above[1].

Lemma 1.2. *$\forall$ f.d. $U_q(\mathfrak{sl}_2)-$modules V and $v \in V$*

$$T(Ev) = (-FK)T(v) \qquad T(Fv) = (-K^{-1}E)T(v) \qquad T(Kv) = K^{-1}T(v)$$

Proof. Suffice to check on $L(q^n)$ by s.s. ∎

Lemma 1.3. *Let $\langle \alpha, \beta \rangle = 0$. Then $E_\beta \cdot T_\alpha(v) = T_\alpha(E_\beta \cdot v)$, similarily for F_β.*

Proof. E_β commutes past everything in the formula for T_α. ∎

Lemma 1.4. *$\forall$ f.d. $U_q(\mathfrak{g})-$modules V and $v \in V$, let $r = -\langle \beta, \alpha^\vee \rangle$, then*

$$T_\alpha(E_\beta v) = \left(\operatorname{ad}_{E_\alpha^{(r)}}(E_\beta)\right) \cdot T_\alpha(v)$$

Theorem 1
Let $u \in U_q(\mathfrak{g})$. If u annihilates all finite-dimensional $U-$modules, then $u = 0$.

Proposition 1.5. *Let α be a simple root. $\forall u \in U$, $\exists!\, u' \in U$ s.t.*

$$T_\alpha(uv) = u' T_\alpha(v) \tag{*}$$

for all f.d. $U-$modules V and $v \in V$. Furthermore $T_\alpha(u) := u'$ is an algebra automorphism of U.

Proof. Existence: For $u_1, u_2 \in U$ suppose we found u_1', u_2' satisfying $(*)$. Then $\implies$ $(u_1 + u_2)' := u_1' + u_2'$ and $(u_1 u_2)' := u_1' u_2'$ also satisfies $(*)$. Thus it suffices to show existence on generators of $U_q(\mathfrak{g})$, but that is exactly the content of the previous 3 lemmas.

Uniqueness: Suppose u', u'' both satisfy $(*)$ for u. Then $(u' - u'')T_\alpha(v) = 0$. But T_α is bijective and thus $(u' - u'')$ annihilate all f.d. $U_q(\mathfrak{g})$ modules and so by Theorem 1 $u' = u''$.

Auto: $T_\alpha : U_q(\mathfrak{g}) \to U_q(\mathfrak{g})$ is an algebra homomorphism by construction above. Using Lemma 1.2 we see that

$$T_\alpha((-K_\alpha^{-1} F_\alpha)v) = -K_\alpha T_\alpha(F_\alpha v) = E_\alpha T_\alpha(v)$$

and so $T_\alpha(-K_\alpha^{-1} F_\alpha) = E_\alpha$. Similar manipulations occur using the lemmas above to show surjectivity of T_α. For injectivity, a trick similar to the proof of uniqueness works. ∎

Let $T_i = T_{\alpha_i}, s_i = s_{\alpha_i}$. Explicitly $T_i : U_q(\mathfrak{g}) \to U_q(\mathfrak{g})$ will be

$$T_i(K_\mu) = K_{s_i(\mu)}, \qquad T_i(E_i) = -F_i K_i, \qquad T_i(F_i) = -K_i^{-1} E_i$$

$$T_i(E_j) = \sum_{k=0}^{-a_{ji}} (-1)^k q_i^{-k} E_i^{(r-k)} E_j E_i^{(k)}$$

Theorem 1.6 (Lusztig)**.** *$\langle T_i \rangle_{\alpha_i \in \Pi}$ satisfy the braid relations for B_W.*

[1] $\langle \mu, \alpha^\vee \rangle$ is also the length of the α string through V_μ.

1.2 PBW Bases for $U_q(\mathfrak{g}^+)$

Fix a total ordering $\beta_1 < \ldots < \beta_M$ on Φ^+ where $M = |\Phi^+|$. Then in the classical setting

$$\left\{ e_{\beta_1}^{a_1} \ldots e_{\beta_M}^{a_m} | a_i \in \mathbb{Z}^{\geq 0} \right\}$$

gives a basis for $U(\mathfrak{g}^+)$. In the quantum setting we only have E_α where α is a simple root. This is where the operators T_α come in. First for any $w \in W$, let $\underline{w} = s_{i_1} s_{i_2} \ldots s_{i_k}$ be a reduced expression for w. Then set

$$T_w := T_{i_1} \ldots T_{i_k}$$

This is well defined byTheorem 1.6 and Matsomoto's theorem.

Lemma 1.7. *Let $\underline{w} = s_{i_1} s_{i_2} \ldots s_{i_k}$ be a reduced expression. Then*

$$\beta_1 = \alpha_{i_1}, \quad \beta_2 = s_{i_1}(\alpha_{i_2}), \quad \beta_3 = (s_{i_1} s_{i_2})(\alpha_{i_3}), \quad \ldots, \quad \beta_k = (s_{i_1} s_{i_2} \ldots s_{i_{k-1}})(\alpha_{i_k})$$

are k distinct positive roots. In fact they are exactly the positive roots γ s.t. $w^{-1}\gamma < 0$.

Corollary 1.8. *Let $w = w_0$, then the above procedure gives all the positive roots from the simple roots.*

Theorem 2 (Lusztig)
Fix a reduced expression $\vec{i} = s_{i_1} \ldots s_{i_M}$ for w_0. Define

$$\begin{aligned} E_{\vec{i}:\beta_1} &:= E_{i_1} \\ E_{\vec{i}:\beta_2} &:= T_{i_1}(E_{i_2}) \\ E_{\vec{i}:\beta_3} &:= T_{i_1} T_{i_2}(E_{i_3}) \\ &\vdots \end{aligned}$$

and set

$$B_{\vec{i}} = \left\{ E_{\vec{i}:\beta_1}^{(a_1)} E_{\vec{i}:\beta_2}^{(a_2)} \cdots E_{\vec{i}:\beta_M}^{(a_M)} \middle| a_i \in \mathbb{Z}^{\geq 0} \right\}$$

Then $B_{\vec{i}}$ is a (PBW) basis for $U_q(\mathfrak{g}^+)$

Warning. $B_{\vec{i}}$ really depends on $\vec{i}$. Take $\mathfrak{g} = \mathfrak{sl}_3$ with $\Pi = \{\alpha_1, \alpha_2\}$. Then $w_0 = \vec{a} = s_1 s_2 s_1$, $w_0 = \vec{b} = s_2 s_1 s_2$. Now notice

$$B_{\vec{a}} = \left\{ E_1, E_1 E_2 - q^{-1} E_2 E_1, E_2 \right\} \qquad B_{\vec{b}} = \left\{ E_2, E_2 E_1 - q^{-1} E_1 E_2, E_1 \right\}$$

2 The Super Case

In the classical case we can choose any set of simple roots Π and $U_q(\mathfrak{g})$ will have the same presentation. In the $\mathfrak{sl}_3$ example, we could take $\Pi = \{\epsilon_1 - \epsilon_2, \epsilon_2 - \epsilon_3\}$ or $\Pi' = \{\epsilon_1 - \epsilon_3, \epsilon_3 - \epsilon_2\}$ and the presentation for $U_q(\mathfrak{g})$ will be the same. [Draw on matrices]

Now interpret Π, Π' as roots for $\mathfrak{gl}(2|1)$. The corresponding Dynkin diagrams will be

$$D(\Pi) = \bigcirc\!\!\longrightarrow\!\!\!\!\!-\!\!\otimes \qquad D(\Pi') = \otimes\!\!\longrightarrow\!\!\!\!\!-\!\!\otimes$$

As a result, we have that

$$U_q^{\Pi}(\mathfrak{gl}(2|1)) = \mathbb{C}(q)\left\langle E_1, E_2, \dots \right\rangle \Big/ (E_1^2E_2 - (q+q^{-1})E_1E_2E_1 + E_2E_1^2,\ E_2^2, \dots)$$
$$U_q^{\Pi'}(\mathfrak{gl}(2|1)) = \mathbb{C}(q)\left\langle E_1', E_2', \dots \right\rangle \Big/ ((E_1')^2, (E_2')^2, \dots)$$

These two algebras are actually isomorphic but it's not clear at the moment why. To remedy this we will introduce algebras $U_q(C_\Pi) \cong U_q(\mathfrak{gl}(m|n))$ for every Cartan matrix corresponding to a choice of simple roots Π.

Definition 2.1 (Super Cartan Matrices)**.** *Let $\mathfrak{g}$ be a basic lie superalgebra and let Π be a choice of simple roots for $\mathfrak{g}$. For $\alpha_i \in \Pi$, let $h_i = [e_{\alpha_i}, f_{\alpha_i}]_s$. Define*

$$C_\Pi = (c_{ij}) = (\alpha_i(h_j))$$

Remark. For $\mathfrak{gl}(m|n)$, $\alpha_i(h_j) = (\alpha_i, \alpha_j)_s$.

Definition 2.2 (Super Dynkin Diagram)**.** *Given a super Cartan Matrix C_Π, the Dynkin diagram $D(\Pi)$ will be the same formula as always except we draw a dashed line between i and i if $c_{ij} > 0$, $i \neq j$.*

Example 1. For $\mathfrak{gl}(3|1)$ we have

Π	$\{\epsilon_1 - \epsilon_2, \epsilon_2 - \epsilon_3, \epsilon_3 - \delta_4\}$	$\{\epsilon_1 - \epsilon_2, \epsilon_2 - \delta_4, \delta_4 - \epsilon_3\}$
C_Π	$\begin{pmatrix} 2 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 0 \end{pmatrix}$	$\begin{pmatrix} 2 & -1 & 0 \\ -1 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}$
$D(\Pi)$	$\bigcirc$ —— $\bigcirc$ —— $\otimes$	$\bigcirc$ —— $\otimes$ - - - - - - $\otimes$

Remark. Let α be an odd isotropic root for $\mathfrak{gl}(m|n)$. Then the odd reflection s_α is actually equal to $s_\alpha \in W(\mathfrak{gl}_{m+n})$. In particular the group generated by $W(\mathfrak{gl}(m|n))$ and odd reflections is S_{m+n}. Therefore we have an action of S_{m+n} on the super Cartan matrices. Explicitly, given $\Pi = \left\{\alpha_1^\Pi, \dots, \alpha_{m+n-1}^\Pi\right\}$ and C_Π, $s_i(C_\Pi) = C_{s_i(\Pi)}$ where $s_i = s_{\alpha_i^\Pi} \in S_{m+n}$. Note that as Π changes the definition of s_i changes as well.

For example, let Π, Π' be the LHS, RHS respectively above. Then $s_3(C_\Pi) = C_{s_{\epsilon_3 - \delta_4}(\Pi)} = C_{\Pi'}$, and $s_1(C_\Pi) = s_2(C_\Pi) = \Pi$. In general, $W(\mathfrak{gl}(m|n)) \subset W(\mathfrak{gl}_{m+n})$ does not change C_Π.

Definition 2.3. *Given $C = C_\Pi$ for $\mathfrak{gl}(m|n)$, let $U_q(C)$ be the $\mathbb{Z}_2$ graded associative $\mathbb{C}(q)$ algebra with generators $E_{C,i}, F_{C,i}, K_{C,i}^{\pm 1}$ with parity $p(E_{C,i}) = p(F_{C,i}) = p(i) = p(\alpha_i)$, $p(K_{C,i}) = 0$ satisfying the relations (We drop C for convenience)*

$$E_i^2 = 0 \qquad \textit{if } c_{ii} = 0 \quad (1)$$
$$E_iE_j = (-1)^{p(i)p(j)}E_jE_i \qquad \textit{if } i \not\sim j \quad (2)$$
$$E_i^2E_j + E_jE_i^2 = (q+q^{-1})E_iE_jE_i \qquad \textit{if } i \sim j \textit{ and } c_{ii} \neq 0 \quad (3)$$
$$\begin{aligned}[2]E_jE_iE_kE_j &= (-1)^{p(k)p(i)+p(i)}E_jE_kE_jE_i + (-1)^{p(k)p(i)+p(k)}E_kE_jE_iE_j \qquad \textit{if } i \sim j \sim k \textit{ and } c_{jj} = 0 \quad (4)\\ &+ (-1)^{p(k)}E_jE_iE_jE_k + (-1)^{p(i)}E_iE_jE_kE_j \qquad \textit{and } c_{ij} \neq c_{jk}\end{aligned}$$
$$\vdots \qquad (5)$$

2.1 The $U_q(\mathfrak{gl}(m|1))$ Case

From now on only work with $U_q(\mathfrak{gl}(m|1))$.

Theorem 3 (Clark)
Let C be a super Cartan matrix for $U_q(\mathfrak{gl}(m|1))$ and set $D = s_i(C)$. Then define $T_i^s : U_q(C) \to U_q(D)$ as

$$T_i^s(E_{C,j}) = \begin{cases} -F_{D,i}K_{D,i} & \text{if } j = i \\ E_{D,i}E_{D,j} - (-1)^{p_D(i)p_D(j)} q^{D_{ij}} E_{D,j}E_{D,i} & \text{if } j \sim i \\ E_{D,j} & \text{if } j \not\sim i \end{cases}$$

We omit the definition for the other generators. Then T_i^s is a $\mathbb{Z}_2-$algebra isomorphism.

Proof. We check some relations namely Eq. (1). Let $C_{jj} = 0$ so that $E_{C,j}^2 = 0$
Case 1: $i \not\sim j$ We then have that $D_{jj} = 0$ as well since s_i only changes the nodes adjacent to it. Thus

$$T_i^s(E_{C,j})^2 = E_{D,j}^2 = 0$$

Case 2: $i \sim j$. $s_i(\alpha_i) = -\alpha_i \implies p_D(i) = p_C(i)$ while since $s_i(\alpha_j) = \alpha_j + \alpha_i$, we see that

$$p_D(j) = p_C(j) + p_C(i) = 1 + p_D(i) \implies i \text{ and } j \text{ have different parity}$$

In other words, the effect of s_i on the Dynkin diagram locally looks like

$$\underset{j}{\otimes}\rule[0.5ex]{4em}{0.4pt}\underset{i}{\bigcirc} \xrightarrow{s_i} \underset{j}{\otimes}\rule[0.5ex]{4em}{0.4pt}\underset{i}{\bigcirc}$$

$$\text{or} \quad \underset{j}{\otimes}\rule[0.5ex]{4em}{0.4pt}\underset{i}{\otimes} \xrightarrow{s_i} \underset{j}{\bigcirc}\rule[0.5ex]{4em}{0.4pt}\underset{i}{\otimes}$$

We check the first case (so $E_j^2 = 0$) and also assume $D_{ij} = 1$ [Clark does $D_{ij} = -1$] We compute

$$\begin{aligned} T_i^s(E_{C,j})^2 &= (E_{D,i}E_{D,j} - qE_{D,j}E_{D,i})^2 \\ &= E_iE_jE_iE_j - qE_iE_j^2E_i - qE_jE_i^2E_j + q^2E_jE_iE_jE_i \\ &\overset{\text{Eq. (3)}}{=} \frac{E_jE_i^2E_j}{(q+q^{-1})} - qE_jE_i^2E_j + \frac{q^2E_jE_i^2E_j}{(q+q^{-1})} = 0 \end{aligned}$$

■

Proposition 2.4 (Clark)**.** *The T_i^s satisfy braid relations of type A between appropriate $U_q(C)$, i.e. if $i \not\sim j$, given a super Cartan matrix B, let $C = s_i(B), D = s_j(C)$, then as maps $U_q(B) \to U_q(D)$ $T_i^sT_j^s = T_j^sT_i^s$, and similarly with $i \sim j$.*

Theorem 4 (Clark)
Fix Π for $\mathfrak{gl}(m|1)$ and let $C = C_\Pi$. Fix a reduced expression $\vec{i} = s_{i_1} \dots s_{i_K}$ for $w_0 \in S_{m+1}$. Define $\beta_t^\Pi = s_{i_1} \cdots s_{i_{t-1}}(\alpha_{i_t}^\Pi)$ and let $C_{\vec{i},t} = s_{t-1} \cdots s_{i_1}(C)$ (so $C_{\vec{i},1} = C$). Finally let

$$\begin{aligned} E_{\vec{i}:\beta_1^\Pi} &:= E_{C,i_1} \\ E_{\vec{i}:\beta_2^\Pi} &:= T_{i_1}^s(E_{C_{\vec{i},2},i_2}) \\ &\vdots \\ E_{\vec{i}:\beta_t^\Pi} &:= T_{i_1}^s \dots T_{i_{t-1}}^s(E_{C_{\vec{i},t},i_t}) \\ &\vdots \end{aligned}$$

and set

$$B_{\vec{i}}^\Pi = \left\{ E_{\vec{i}:\beta_1^\Pi}^{(a_1)} E_{\vec{i}:\beta_2^\Pi}^{(a_2)} \cdots E_{\vec{i}:\beta_L^\Pi}^{(a_L)} \,\middle|\, a_i \in \mathbb{Z}^{\geq 0}, a_s < 2 \text{ if } p(\beta_s^\Pi) = 1 \right\}$$

Then $B_{\vec{i}}^\Pi$ is a (PBW) basis for $U_q^+(C)$.

Remark. Because $E_{C_{\vec{i},t},i_t} \in U_q(C_{\vec{i},t}) = U_q(s_{t-1} \dots s_{i_1}(C))$ we see that

$$T_{i_1}^s \dots T_{i_{t-1}}^s(E_{C_{\vec{i},t},i_t}) \in U_q((s_{i_1} \dots s_{i_{t-1}})(s_{i_{t-1}} \dots s_{i_1})(C)) = U_q(C)$$

The miracle is that it's in fact in $U_q^+(C)$.